

**Radiological Control Technician Training**  
**Fundamental Academic Training Instructor's Guide Phase I**



**Coordinated and Conducted**  
**for**  
**Office of Environment, Safety & Health**  
**U.S. Department of Energy**

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**Course Title:** Radiological Control Technician  
**Module Title:** Basic Mathematics and Algebra  
**Module Number:** 1.01

**Objectives:**

- 1.01.01 Add, subtract, multiply, and divide fractions.
- 1.01.02 Add, subtract, multiply, and divide decimals.
- 1.01.03 Convert fractions to decimals and decimals to fractions.
- 1.01.04 Convert percent to decimal and decimal to percent.
- 1.01.05 Add, subtract, multiply, and divide signed numbers.
- 1.01.06 Add, subtract, multiply, and divide numbers with exponents.
- 1.01.07 Find the square roots of numbers.
- 1.01.08 Convert between numbers expressed in standard form and in scientific notation.
- 1.01.09 Add, subtract, multiply, and divide numbers expressed in scientific notation.
- 1.01.10 Solve equations using the "Order of Mathematical Operations."
- 1.01.11 Perform algebraic functions.
- 1.01.12 Solve equations using common and/or natural logarithms.

**References:**

- 1. DOE-HDBK-1014/1-92 (June 1992) "Mathematics: Volume 1 of 2"; DOE Fundamentals Handbook Series.
- 2. "Algebra Structure and Method: Book 1"; Brown, Richard G., et al; Houghton Mifflin Co.; 1990.
- 3. "Essential Mathematics for College Students"; Steffenson, Arnold R., Johnson, L. Murphy; Scott, Foresman and Co.; 1988.
- 4. "College Physics"; sixth ed.; Schaum, Daniel; McGraw-Hill; 1961.

**Instructional Aids:**

- 1. Overheads
- 2. Overhead projector/screen
- 3. Chalkboard/whiteboard
- 4. Lessons learned

**I. MODULE INTRODUCTION**

## A. Self Introduction

1. Name
2. Phone number
3. Background
4. Emergency procedure review

## B. Motivation

Radiological control operations frequently require the RCT to use arithmetic and algebra to perform various calculations. These include scientific notation, unit analysis and conversion, radioactive decay calculations, dose rate/distance calculations, shielding calculations, and stay-time calculations. A good foundation in mathematics and algebra is important to ensure that the data obtained from calculations is accurate. Accurate data is crucial to the assignment of proper radiological controls.

## C. Overview of Lesson

1. Fractions
2. Decimals
3. Percent
4. Signed Numbers
5. Exponents
6. Square Roots
7. Scientific Notation
8. Order of Mathematical Operations
9. Algebra
10. Logarithms

## D. Introduce Objectives

O.H.: Objectives

**II. MODULE OUTLINE****A. Symbols for Basic Operations**

1. The four basic mathematical operations are addition, subtraction, multiplication, and division. Furthermore, it is often necessary to group numbers or operations using parentheses or brackets. In writing problems in this course the notation shown in Table 1 is used to denote the operation to be performed on the numbers.

See Table 1. - "Symbols for Basic Mathematical Operations"

**B. Fractions**

1. Whole numbers consist of the normal counting numbers and zero.  
  
{e.g., 0, 1, 2, 3, 4...}
2. A fraction is part of a whole number. It is simply an expression of a division of two whole numbers. A fraction is written in the format:

$$\frac{a}{b} \text{ or } a/b$$

- a. The number above the bar  $a$  is called the numerator and the number below the bar  $b$  is called the denominator. A *proper fraction* is a fraction in which the number in the numerator is less than the number in the denominator. If the numerator is greater than the denominator then it is an improper fraction. For example,  $\frac{1}{2}$  and  $\frac{1}{4}$  are proper fractions, while,  $\frac{25}{5}$ ,  $\frac{15}{7}$ , or  $\frac{61}{27}$  are improper fractions.
- b. Any whole number can be written as a fraction by letting the whole number be the numerator and 1 be the denominator. For example:

$$5 = \frac{5}{1} \quad 2 = \frac{2}{1} \quad 0 = \frac{0}{1}$$

- c. Five can be written as  $\frac{10}{2}$ ,  $\frac{15}{3}$ ,  $\frac{20}{4}$ , etc. Similarly, the fraction  $\frac{1}{4}$  can be written as  $\frac{2}{8}$ ,  $\frac{3}{12}$ ,  $\frac{4}{16}$ , etc. These are called equivalent fractions. An equivalent fraction is built up, per se, by multiplying the numerator and the denominator by the same non-zero number. For example:

$$\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8} \quad \frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20}$$

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- d. A fraction is reduced by dividing the numerator and the denominator by the same non-zero number. For example:

$$\frac{12}{18} = \frac{12 \div 2}{18 \div 2} = \frac{6}{9} \qquad \frac{6}{9} = \frac{6 \div 3}{9 \div 3} = \frac{2}{3}$$

- e. A fraction is reduced to lowest terms when 1 is the only number that divides both numerator and denominator evenly. This is done by finding the greatest common multiple between the numerator and denominator. In the previous example, two successive reductions were performed. For the fraction  $12/18$ , the greatest common multiple would be 6, or  $(2 \times 3)$ , which results in a reduction down to a denominator of 3.
- f. A whole number written with a fraction is called a mixed number. Examples of mixed numbers would be  $1\frac{1}{2}$ ,  $3\frac{3}{4}$ ,  $5\frac{3}{4}$ , etc. A mixed number can be simplified to a single improper fraction using the following steps:

- 1) Multiply the whole number by the denominator of the fraction.
- 2) Add the numerator of the fraction to the product in step 1.
- 3) Place the sum in step 2 as the numerator over the denominator.
- 4) Example:

$$5\frac{3}{4} = \frac{(5 \times 4) + 3}{4} = \frac{23}{4}$$

### 3. Adding and Subtracting Fractions

#### a. Fractions with the Same Denominator

- 1) To add two fractions that have the same denominator:
  - a) Add the numerators
  - b) Place the sum of step 1 over the common denominator

- c) Reduce fraction in step 2 to lowest terms (if necessary)

- 2) Example:

$$\frac{1}{5} + \frac{3}{5} = \frac{1+3}{5} = \frac{4}{5}$$

- 3) Subtraction of two fractions with the same denominator is accomplished in the same manner as addition. For example:

$$\frac{5}{8} - \frac{3}{8} = \frac{5-3}{8} = \frac{2}{8} = \frac{2 \div 2}{8 \div 2} = \frac{1}{4}$$

- b. Fractions with Different Denominators

- 1) To add two fractions with different denominators requires that the fractions be built up so that they have the same denominator. This is done by finding the lowest common denominator. Once a common denominator is obtained, the rules given above for the same denominator apply.
- 2) For example,  $1/3 + 2/5$ . The fraction  $1/3$  could be built up to  $2/6$ ,  $3/9$ ,  $4/12$ ,  $5/15$ ,  $6/18$ ,  $7/21$ , etc. The fraction  $2/5$  could be built up to  $4/10$ ,  $6/15$ ,  $8/20$ ,  $10/25$ , etc. The lowest common denominator for the two fractions would be 15. The problem would be solved as follows:

$$\frac{1}{3} + \frac{2}{5} = \frac{1 \times 5}{3 \times 5} + \frac{2 \times 3}{5 \times 3} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}$$

- 3) Subtraction of fractions with different denominators is accomplished using the same steps as for addition. For example:

$$\frac{3}{4} - \frac{2}{3} = \frac{3 \times 3}{4 \times 3} - \frac{2 \times 4}{3 \times 4} = \frac{9}{12} - \frac{8}{12} = \frac{1}{12}$$

#### 4. Multiplying and Dividing Fractions

- a. Multiplication of fractions is much easier than addition and subtraction, especially if the numbers in the

numerators and denominators are small. Fractions with larger numerators and/or denominators may require additional steps. In either case, the product of the multiplication will most likely need to be reduced in order to arrive at the final answer. To multiply fractions:

- 1) Multiply the numerators.
- 2) Multiply the denominators.
- 3) Place product in step 1 over product in step 2.
- 4) Reduce fraction to lowest terms.
- 5) Example:

$$\frac{5}{6} \times \frac{3}{4} = \frac{5 \times 3}{6 \times 4} = \frac{15}{24} = \frac{15 \div 3}{24 \div 3} = \frac{5}{8}$$

- b. A variation on the order of the steps to multiply fractions is to factor the numerators and denominators first, reduce and cancel, and then multiply. For example:

$$\frac{3}{8} \times \frac{20}{9} = \frac{3}{2 \times 2 \times 2} \times \frac{2 \times 2 \times 5}{3 \times 3} = \frac{3 \times 2 \times 2 \times 5}{2 \times 2 \times 2 \times 3 \times 3} = \frac{5}{2 \times 3} = \frac{5}{6}$$

## 5. Reciprocals

- a. Two numbers whose product is 1 are called reciprocals, or multiplicative inverses. For example:

1) 5 and  $\frac{1}{5}$  are reciprocals because  $5 \times \frac{1}{5} = 1$

2)  $\frac{4}{5}$  and  $\frac{5}{4}$  are reciprocals because  $\frac{4}{5} \times \frac{5}{4} = 1$

3) 1 is its own reciprocal because  $1 \times 1 = 1$

4) 0 has no reciprocal because 0 times any number is 0 not 1

- b. The symbol for the reciprocal, or multiplicative inverse, of a non-zero real number  $a$  is  $\frac{1}{a}$ . Every real

number except 0 has a reciprocal. Therefore, for every non-zero real number  $a$ , there is a unique real number

$\frac{1}{a}$  such that

$$a \times \frac{1}{a} = 1$$

- c. Now, look at the following product:

$$(ab)\left(\frac{1}{a} \times \frac{1}{b}\right) = \left(a \times \frac{1}{a}\right)\left(b \times \frac{1}{b}\right) = 1 \times 1 = 1$$

#### 6. Relationship of multiplication to division

- a. The operation of division is really just inverted multiplication (reciprocals). Notice from the above examples the reciprocal of a fraction is merely "switching" the numerator and denominator. The number 5 is really  $5/1$ , and the reciprocal of 5 is  $1/5$ . Likewise, the reciprocal of  $2/3$  is  $3/2$ .
- b. Fractions are a division by definition. Division of fractions is accomplished in two steps:
- 1) Invert the second fraction, i.e., change it to its reciprocal, and change the division to multiplication.
  - 2) Multiply the two fractions using the steps stated above.
  - 3) Examples:

$$\frac{1}{2} = 1 \div \frac{2}{3} = 1 \times \frac{3}{2} = \frac{3}{2}$$

#### C. Decimals

1. A decimal is another way of expressing a fraction or mixed number. It is simply the numerical result of division (and fractions are division). Recall that our number system is based on 10 ("deci" in "decimal" means ten) and is a place-value system; that is, each digit {i.e., 0, 1, 2, 3, 4, 5, 6, 7, 8,

Objective 1.01.02

See Fig. 1 - "Decimal Places"

9} in a numeral has a particular value determined by its location or place in the number. For a number in decimal notation, the numerals to the left of the decimal point comprise the whole number, and the numerals to the right are the decimal fraction, (with the denominator being a power of ten).

2. For example, the numeral 125.378 (decimal notation) represents the expanded numeral:

$$100 + 20 + 5 + \frac{3}{10} + \frac{7}{100} + \frac{8}{1000}$$

3. If this numeral were written as a mixed number, we would write it as:

$$125\frac{378}{1000}$$

4. Addition and Subtraction of Decimals

- a. In order to add or subtract decimals use the following steps:

- 1) Arrange the numbers in a column so that the decimal points are aligned.
- 2) Add or subtract in columns from right to left. (Additional zeros may need to be added to the right of the subtrahend.)
- 3) Place the decimal point in the answer in line with the other decimal points.
- 4) Examples:

$$\begin{array}{r} 21.3 \\ +4.2 \\ \hline 25.5 \end{array} \qquad \begin{array}{r} 654.200 \\ -26.888 \\ \hline 627.312 \end{array}$$

5. Multiplying Decimals

- a. To multiply decimal numbers, do the following:

- 1) Multiply the numbers as if there were no decimal points.

Subtrahend - a number that is to be subtracted from the minuend).

- 2) Count the number of decimal places in each number and add them together.
- 3) Place the decimal point in the product so that it has the same number of decimal places as the sum in step 2.
- 4) Examples:

$$\begin{array}{r}
 5.28 \\
 \times 3.7 \\
 \hline
 3696 \\
 1584 \\
 \hline
 19.536 \\
 \leftarrow\leftarrow\leftarrow
 \end{array}$$

$$\begin{array}{r}
 0.04 \\
 \times 0.957 \\
 \hline
 028 \\
 020 \\
 036 \\
 000 \\
 \hline
 0.03828 \\
 \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow
 \end{array}$$

#### 6. Division of Decimals

- a. The steps for division of decimals are as follows:

- 1) Move the decimal point of the divisor to the right until it becomes a whole number.
- 2) Move the decimal point of the dividend to the right the same number of places moved in step 1.
- 3) Divide the numbers as if they were whole numbers.
- 4) Example:

Divisor - The number by which a dividend is divided.

Dividend - a number to be divided.

$$\begin{array}{r}
 7 \div 0.25 \\
 \underline{28} \\
 0.25 \overline{)7.00} \\
 \underline{50} \\
 200 \\
 \underline{200} \\
 0
 \end{array}$$

## 7. Decimal Forms

- a. As we have learned, decimals are the result of division (or a fraction). When the remainder of the division is zero, the resulting decimal is called a terminating or finite decimal. For example, fractions like  $2/5$ ,  $3/8$ , and  $5/6$  will all result in finite decimals. These fractions and the resulting decimals are known as rational numbers.
- b. On the other hand, fractions like  $1/3$ ,  $2/7$ ,  $5/11$ , and  $7/13$  result in a non-terminating or infinite decimal. For example,  $2/7$  results in the decimal  $0.285714286 \dots$ , the dots meaning that the decimal continues without end. These numbers are known as irrational numbers. Note that even though irrational numbers are non-terminating, some (e.g.,  $1/3$  and  $5/11$ ) are repeating or periodic decimals because the same digit or block of digits repeats unendingly. For example:

$$\frac{1}{3} = 0.3333\dots \qquad \frac{5}{11} = 0.454545\dots$$

- c. A bar is often used to indicate the block of digits that repeat, as shown below:

$$\frac{1}{3} = 0.3 \qquad \frac{5}{11} = 0.\overline{45}$$

## D. Fraction to Decimal Conversion

To convert a fraction to a decimal we simply perform the operation of division that the fraction represents. For example, the fraction  $3/4$  represents "3 divided by 4," and would be converted as follows:

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$$\begin{array}{r} 0.75 \\ 4 \overline{)3.00} \\ \underline{28} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

## E. Percent

Objective 1.01.04

1. Percentage is a familiar and widely used concept for expressing common and decimal fractions. Most people know the meaning of terms such as 100 percent and 50 percent. The word percent actually means "out of a hundred." (Consider that there are 100 "cents" in a dollar, and that a "century" is 100 years.) A percent is simply a fraction whose denominator is 100. Thus, 50 percent means 50/100, or 0.50, while 25 percent means 25/100, or 0.25.

2. Percent is abbreviated by the symbol %. So, 75 percent is written 75%.

## 3. Converting Decimal to Percent

- a. A decimal fraction is changed to a percent by moving the decimal point two places to the right and adding a percent sign. For example,  $\frac{1}{8}$  equals 0.125. Therefore:

$$\frac{1}{8} = 0.\underline{1}\underline{2}5 = 12.5\%$$

- b. A percent is changed to a common fraction by omitting the percent sign, placing the number over 100, and reducing the resulting fraction if possible. For example, 32% equals 32/100 which reduces to 8/25. When the percent consists of a mixed decimal number with a percent sign, the resulting fraction will contain a mixed decimal numerator. This can be changed to a whole number by multiplying the numerator and the denominator by 10, 100, 1,000, etc. For example:

$$40.25\% = \frac{40.25}{100} = \frac{40.25 \times 100}{100 \times 100} = \frac{4025}{10,000} = \frac{805}{2000}$$

c. Percentage is most frequently used to indicate a fractional part. Thus 20% of the total power output for 75% of the employees refer to fractional parts of some total number. To perform arithmetic operations with a percent, it is normally changed to a common or decimal fraction. Usually, a decimal fraction is more convenient.

#### 4. Converting Percent to Decimal

a. A percent is changed to a decimal fraction by omitting the percent sign and moving the decimal point two places to the left. For example:

$$48\% = 0.\underline{4}8$$

b. Thus, 92% equals 0.92, 8% equals 0.08, and so on.

#### F. Signed Numbers

1. The numbers that are used to quantify the number of objects in a group, the "counting numbers," are always positive numbers; that is, they are always greater than zero. However, there are many occasions when negative numbers (numbers less than zero) must be used. These numbers arise when we try to describe measurement in a direction opposite to the positive numbers. For example, if we assign a value of +3 to a point which is 3 feet above the ground, what number should be assigned to a point which is 3 feet below the ground? Perhaps the most familiar example of the use of negative numbers is the measurement of temperature, where temperatures below an arbitrary reference level are assigned negative values.
2. Every number has a sign associated with it. The plus (+) sign indicates a positive number, whereas the minus (–) sign indicates a negative number. When no sign is given, a plus sign (+) is implied. The fact that the plus and minus signs are also used for the arithmetic operations of addition and subtraction should not be a cause for confusion, for we shall see that they have equivalent meanings.
3. Every number has an absolute value, regardless of its sign. The absolute value indicates the distance from zero, without regard to direction. The number 5 is 5 units from zero, in the positive direction. The number –5 is also 5 units from zero, but in the negative direction. The absolute value of each of these numbers is 5. The absolute

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value of a number is indicated by a pair of vertical lines enclosing the number:  $|5|$ .

#### 4. Operations with Signed Numbers

- a. The arithmetic operations of addition, subtraction, multiplication, and division of signed numbers can be more easily visualized if the numbers are placed on a number line. The positive numbers are greater than zero, and they lie to the right of zero on the number line. The negative numbers are less than zero, and lie to the left of zero on the number line.
- b. The number line extends an infinite distance in each direction and therefore includes all numbers. The process of addition can be considered as counting in one direction or the other from a starting point on the number line. For example, let us add  $1 + 2$ . We locate  $+1$  on the number line and then count 2 units to the right, since we are adding  $+2$ . The result will be  $+3$ . To illustrate further, let us add  $+2$  and  $-4$ . We first locate  $+2$  on the number line and then count 4 units to the left. We end up at  $-2$ .
- c. The number line is useful for illustrating the principles of addition, but it clearly would be inconvenient to use in the case of large numbers. Consequently, the following rules were developed to govern the addition process:

See Fig. 2 - "Number Line"

#### 5. Adding and Subtracting Signed Numbers

- a. To add two numbers with the same signs, add their absolute values and attach the common sign. For example:

$$(-3) + (-2) = -5$$

- b. To add two numbers with opposite signs, find the difference of their absolute values, then attach the sign of the original number which had the greater absolute value. For example:

$$(-2) + 3 = 1$$

- c. Notice that  $-3$  and  $+3$  are the same distance but in opposite directions from  $0$  on the number line. What happens when you add two numbers like  $3$  and  $-3$ ?

$$3 + (-3) = 0$$

$$-7 + 7 = 0$$

- d. If the sum of two signed numbers is  $0$ , the numbers are called additive inverses or opposites. For example:

$$7 - 3 = 4 \text{ is the same as: } 7 + (-3) = 4$$

$$8 - 2 = 6 \text{ is the same as: } 8 + (-2) = 6$$

- e. It can be seen that subtracting a number is equivalent to adding its additive inverse or opposite.
- f. To subtract a signed number, add its opposite or additive inverse. In other words, change the subtraction symbol to addition and change the sign of the second signed number. For example:

$$5 - (-8) = 5 + (+8) \leftarrow \text{add } +8$$

(Answer = 13)

$$6 - 11 = 6 + (-11) \leftarrow \text{add } -11$$

(Answer = -5)

$$-4 - (-7) = -4 + (+7) \leftarrow \text{add } +7$$

(Answer = 3)

## 6. Multiplying and Dividing Signed Numbers

- a. The product of two numbers with like signs is a positive number. The product of two numbers with unlike signs is a negative number. In symbols:

$$(+ ) \times (+ ) = (+ )$$

$$(+ ) \times (- ) = (- )$$

$$(- ) \times (- ) = (+ )$$

$$(- ) \times (+ ) = (- )$$

- b. Examples:

$$(-4) \times (-3) = (+12)$$

$$(-4) \times (+3) = (-12)$$

- c. The division of numbers with like signs gives a positive quotient. The division of numbers with unlike signs gives a negative quotient. In symbols:

$$(+)/(+) = (+)$$

$$+)(- = (-)$$

$$-)/(- = (+)$$

$$-)/(+ = (-)$$

- d. Examples:

$$(-24)/(-6) = (+4)$$

$$(-24)/(+6) = (-4)$$

- e. Remember that multiplication is really a short form of addition. When we say  $+4 \times (-3)$ , we are adding the number  $-3$  four times, that is,  $(-3) + (-3) + (-3) + (-3) = -12$ . Also, since division is a short form of subtraction, when we say  $-24 \div (-6)$ , we subtract the number  $-6$  from the number  $-24$  four times in order to reach 0, i.e.,  $-24 - (-6) - (-6) - (-6) - (-6) = 0$ . Although we could repeat the process for the multiplication and division of any signed numbers, usage of the two rules will produce equivalent results.

## G. Exponents

1. An exponent is a small number placed to the right and a little above another number, called the base, to show how many times the base is to be multiplied by itself. Thus,  $3^4$  (read "three to the fourth power") means 3 used as a factor four times or  $3 \times 3 \times 3 \times 3$ . In this case, 4 is the exponent, and 3 is the base.
2. In general, if  $b$  is any real number and  $n$  is any positive integer, the  $n$ th power of  $b$  is written  $b^n$  (where  $b$  is the base and  $n$  is the exponent) and is read as " $b$  to the  $n$ th power." This tells you that  $b$  is used as a factor  $n$  times.
3. Thus,  $5^2$  is called "5 raised to the second power" (or 5 "squared"), and  $2^3$  is called "2 raised to the third power" (or 2 "cubed"). When no exponent is shown for a number or no power is indicated, the exponent or power is understood to be 1. Thus, 7 is the same as  $7^1$ . Any number raised to the power of zero equals one; e.g.,  $7^0 = 1$ .

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Normally, exponents of zero and one are not left as the final value, but are changed to the simpler form of the base.

4. Exponents can be expressed as integers, as in the examples above, or as fractions or decimals such as  $9^{1/2}$  or  $10^{3.2}$ . They may also be positive or negative.
5. Exponents and powers are particularly useful in mathematics not only because they shorten the writing of mathematical expressions, but also because they simplify many mathematical operations. However, there are several special rules which govern mathematical operations involving numbers with exponents.

#### 6. Addition and Subtraction

- a. The addition or subtraction of numbers with exponents can be performed only if both the bases and the exponents of the numbers are the same. When the bases of the exponents are different, the multiplication indicated by the exponent must be performed and the numbers then added or subtracted. Thus,  $2^5$  and  $2^4$  cannot be added directly because their exponents are different. They can be added only by carrying out the indicated multiplication first. Thus,  $2^5$  equals  $2 \times 2 \times 2 \times 2 \times 2$  which equals 32, and  $2^4$  equals  $2 \times 2 \times 2 \times 2$  which equals 16. Therefore,  $2^5 + 2^4$  equals  $32 + 16$ , which equals 48.
- b. When the bases and the exponents are the same, the numbers can be added or subtracted directly. For example:

$$3^5 + 3^5 = 2(3^5) = 2(243) = 486$$

#### 7. Multiplication

- a. The multiplication of numbers with exponents of the same base is performed by adding the exponents. The general form is as follows:

$$(a^m)(a^n) = a^{(m+n)}$$

- b. It is important to remember that the bases of the numbers must be the same before they can be multiplied by adding their exponents. The base of the product is the same as the base of the two factors. Thus,

$$3^2 \times 3^3 = 3^{(2+3)} = 3^5 = 243$$

## 8. Division

- a. The division of numbers with exponents of the same base is performed by subtracting the exponent of the divisor (denominator) from the exponent of the dividend (numerator). The general form is:

$$\frac{a^m}{a^n} = a^{(m-n)}$$

- b. Again, it is important to remember that the bases of the numbers must be the same before they can be divided by subtracting their exponents. The base of the quotient is the same as the base of the number divided. Thus,

$$\frac{2^5}{2^2} = 2^{(5-2)} = 2^3 = 8$$

- c. Division of numbers with exponents can be used to show why any number raised to the power of zero equals one. We know that any fraction in which the numerator equals the denominator can be reduced to 1; e.g.,  $2/2 = 1$ . Similarly:

$$\frac{2^3}{2^3} = 2^{(3-3)} = 2^0 = 1$$

## 9. Exponent Raised to a Power

- a. Raising a number with an exponent to a power is performed by multiplying the exponent by the power. The general form is:

$$(a^m)^n = a^{mn}$$

- b. The base of the answer is the same as the base of the number raised to the power. Thus:

$$(5^2)^3 = 5^{(2 \times 3)} = 5^6 = 15,625$$

## 10. Product Raised to a Power

- a. Raising a product of several numbers to a power is performed by raising each number to the power. The general form is as follows:

$$(ab)^n = a^n b^n$$

- b. Example:

$$[(2)(3)(4)]^2 = (2^2)(3^2)(4^2) = (4)(9)(16) = 576$$

- c. This same result can also be obtained like this:

$$[(2)(3)(4)]^2 = 24^2 = 576$$

## 11. Mixed Product and Exponents Raised to a Power

- a. The same rule can be used to raise a product of several numbers with exponents to a power. The general form looks like this:

$$(A^a B^b C^c)^n = A^{(a \times n)} B^{(b \times n)} C^{(c \times n)}$$

- b. Example:

$$[(2)^4 (3)^3 (4)^2]^2 = (2^{4 \times 2})(3^{3 \times 2})(4^{2 \times 2}) = (2^8)(3^6)(4^4) = (256)(729)(256)$$

## 12. Fraction Raised to a Power

- a. Raising a fraction to a power is performed by raising both numerator and denominator to the power. It should be remembered that with a proper fraction (i.e., numerator is less than the denominator) the resulting number must be less than one. Also, the resulting number will be less than the value of the original fraction. Thus,  $(2/3)^3$  equals  $2^3/3^3$ , which equals  $8/27$ , which is less than one and less than the original fraction,  $2/3$ .

## 13. Negative Exponents and Powers

- a. A negative exponent or power has a special meaning. Any number, except 0, with a negative exponent equals the reciprocal of the same number with the same positive exponent. For example:

$$6^{-2} = \frac{1}{6^2} = \frac{1}{36}$$

- b. The same rules for addition, subtraction, multiplication, division, and raising to a power apply to negative exponents that apply to positive exponents. However, in adding, subtracting, or multiplying the exponents, the rules for signed numbers must also be observed.

#### 14. Fractional Exponents

- a. Fractional exponents are used to represent roots (see next section). The general form of a fractional exponent is  $a^{\frac{m}{n}}$ , which reads "the  $n$ th root of  $a^m$ ." For example,  $a^{1/2}$ , means the square root of  $a^1$ , or  $a$ . In other words,  $a^{1/2} = \sqrt{a}$ .

#### 15. Calculator Method

- a. To raise a number to a power using a scientific calculator, use the following steps:
- 1) Enter the number. (Fractions must first be converted to decimal form.)
  - 2) Press the  $y^x$  key.
  - 3) Enter the power. (Fractions must first be converted to decimal form.)
  - 4) Press the  $=$  key. The number displayed will be the number entered in step 1 raised to the power entered in step 3.

#### H. Square Roots

1. To square a number means to multiply the number by itself, i.e., raise it to the second power. (Consider that a square is two-dimensional.) For example, 2 squared is 4, since  $2 \times 2 = 4$ . The square of 3 is 9, 4 squared is 16, and so on.
2. Just as subtraction "undoes" addition and division "undoes" multiplication, squaring a number can be "undone" by

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finding the square root. The general definition is as follows:

If  $a^2 = b$ , then  $a$  is a square root of  $b$ .

3. Be careful not to confuse the terms square and square root. For example, if  $5^2 = 25$ , this indicates that 25 is the square of 5, and 5 is the square root of 25. To be explicit, we say that it is a perfect square because 5 times itself is 25.
4. All perfect squares other than 0 have two square roots, one positive and one negative. For example, because  $7^2 = 49$  and  $(-7)^2 = 49$ , both 7 and  $-7$  are square roots of 49. The symbol  $\sqrt{\quad}$ , referred to as the radical, is used to write the principal, or positive, square root of a positive number.

$\sqrt{49} = 7$  is read "The positive square root of 49 equals 7."

5. A negative square root is designated by the symbol  $-\sqrt{\quad}$ .

$-\sqrt{49} = -7$  is read "The negative square root of 49 equals  $-7$ ."

6. It is often convenient to use plus-or-minus notation:

$\pm\sqrt{49}$  means the positive or negative square root of 49.

7. Therefore, the rule is that every positive real number  $a$  has two square roots:  $\sqrt{a}$  and  $-\sqrt{a}$ .

8. It follows from the definition of square root that  $(\sqrt{a})^2 = a$ , and that  $\sqrt{a^2} = a$ . Because the square of every real number is either positive or zero, negative numbers do not have square roots in the set of real numbers.

9. Notice that  $\sqrt{4 \cdot 25} = \sqrt{100} = 10$ , and  $\sqrt{4} \cdot \sqrt{25} = 2 \cdot 5 = 10$ . Therefore,  $\sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25}$ . Therefore, in general, we can say:

- a. For any non-negative real numbers  $a$  and  $b$ :

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

b. It also follows that  $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

#### 10. Calculator Method

- a. To calculate the square root of any number using a scientific calculator, follow these steps:
- 1) Enter the number. (Fractions must first be converted to decimal form.)
  - 2) Press the  $\sqrt{x}$  key. The number displayed will be the square root of the number entered in step 1. An alternate method is to press the  $y^x$  key and then type 0.5. This raises the number in step 1 to the power of 0.5, or  $\frac{1}{2}$ .

#### 11. Other roots

- a. For informational purposes only, we mention the fact that other roots may be found for a number. One of these is the cube root. To cube a number means to multiply the number by itself three times, i.e., raise it to the third power. (Consider that a cube is three-dimensional.) For example, 2 cubed is 8, since  $2 \times 2 \times 2 = 8$ . The cube root (or third root) of a number, then, is the number that, when raised to the third power (cubed), equals the first number. The notation for a cube root is  $\sqrt[3]{a}$ .
- b. Note that any root may be taken from a number to "undo" an exponent, such as the fourth or fifth root. The general definition for a root is:

If  $a^n = b$ , then  $a$  is the  $n$ th root of  $b$ .

- c. The notation for the  $n$ th root is  $\sqrt[n]{a}$ . These roots follow the same general rules as the square root.

#### I. Scientific Notation

1. The difficulty in writing very large or very small numbers in the usual manner is that a large number of zeros are required to write these numbers. This difficulty is overcome by using scientific notation, in which integral powers of ten are used instead of a large number of zeros to

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indicate the position of the decimal point. In addition to simplifying the writing of very large or very small numbers, scientific notation clearly identifies the number of significant digits in a number and simplifies arithmetic calculations involving multiplication, division, or raising to a power. For these reasons, it is good practice to write numbers in scientific notation when these operations are involved.

## 2. Converting From Standard Form To Scientific Notation

There are two steps involved in writing a number in scientific notation.

- a. Move the decimal point just to the right of the first significant digit. The first significant digit is the first non-zero digit counting from the left.
- b. Indicate multiplication of the resulting number by a power of ten that makes its value equal to the original value. The power of ten is found by counting the number of places the decimal point was moved from its original position. If counted to the left, the power is positive; if counted to the right, it is negative. For example:

- 1) Suppose you want to express a number such as 700 in scientific notation.

$$\underline{7}00 = 7 \times 10^2$$

- 2) Suppose you want to express 0.0014 in scientific notation.

$$0.\underline{001}4 = 1.4 \times 10^{-3}$$

## 3. Converting from Scientific Notation to Standard Form

- a. To transform from scientific notation to standard form, follow the opposite procedure.

$$1.96 \times 10^5 = 1.\underline{96000} = 196,000$$

$$2.27 \times 10^{-2} = \underline{002}.27 = 0.0227$$

- b. There are two parts of a number written in scientific notation, the significant digits and the power of ten. Thus, in the number  $3.21 \times 10^6$ , 3, 2, and 1 are the significant digits and  $10^6$  is the power of ten.
- c. The ability to clearly see the number of significant digits can be helpful in performing arithmetic calculations. For example, the number of significant digits which should be reported in the product of two numbers can be readily determined if the two numbers are first written in scientific notation.
- d. When numbers are expressed in scientific notation, calculations can be more easily visualized. This is because they involve only numbers between 1 and 10 and positive and negative integral powers of ten which can be treated separately in the calculations using the rules for numbers with exponents.
4. Addition and Subtraction Using Scientific Notation
- a. Addition and subtraction cannot normally be performed directly using scientific notation because they require adding or subtracting digits of equal place value. Thus, when numbers expressed in scientific notation are to be added or subtracted, they must first be converted to forms having equal place value. This is commonly done by expressing them as numbers which are multiplied by the same integral power of ten. The sum or difference of these significant digits, multiplied by their common power of ten, is the sum or difference of the original numbers.
- b. For example:
- $$(3.54 \times 10^5) + (2.51 \times 10^4)$$
- $3.54 \times 10^5$  is first changed to  $35.4 \times 10^4$
- $$\begin{array}{r} 35.4 \times 10^4 \\ + 2.51 \times 10^4 \\ \hline 37.91 \times 10^4 = 3.79 \times 10^5 \end{array}$$
5. Multiplication and Division Using Scientific Notation
- a. Multiplication or division of numbers using scientific notation is performed by multiplying or dividing the

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significant digits and the powers of ten separately. The significant digits are multiplied or divided in the same manner as other mixed decimals. The powers of ten are multiplied or divided by adding or subtracting their exponents using the rules for multiplication and division of numbers with exponents. For example:

$$(2.7 \times 10^2)(3.1 \times 10^{-3}) = (2.7)(3.1) \times (10^2)(10^{-3}) = 8.37 \times 10^{-1}$$

which should be rounded off to  $8.4 \times 10^{-1}$ .

- b. One of the most useful applications of scientific notation is in arithmetic calculations which involve a series of multiplications and divisions. The use of scientific notation permits accurate location of the decimal point in the final answer.
- c. Example: Perform the following calculation using scientific notation:

$$\frac{(219)(0.00204)}{(21.2)(0.0312)}$$

- 1) Write each term in scientific notation:

$$\frac{(2.19 \times 10^2)(2.04 \times 10^{-3})}{(2.12 \times 10^1)(3.12 \times 10^{-2})}$$

- 2) Multiply and divide the significant digits:

$$\frac{(2.19)(2.04)}{(2.12)(3.12)} = \frac{4.46}{6.61} = 0.675$$

- 3) Multiply and divide the powers of ten by adding and subtracting exponents:

$$\frac{(10^2)(10^{-3})}{(10^1)(10^{-2})} = \frac{10^{-1}}{10^{-1}} = 10^{-1+1} = 10^0$$

- 4) Combine the results:

$$0.675 \times 10^0 = 0.675 = 6.75 \times 10^{-1}$$

## 6. "E" Notation

- a. An alternate method for annotating scientific notation is often used by pocket calculators, computers, and some references. The method uses an E in place of the " $\times 10$ ," and the number written after the E is the exponent of 10. The standard and alternate methods for scientific notation are equivalent and can be converted from one form to another without a change in value. The examples below use both methods in equivalent expressions:

$$3.79 \times 10^5 = 3.79E5$$

$$4.02 \times 10^{-6} = 4.02E-6$$

$$5.89 \times 10^0 = 5.89E0$$

## 7. Using "E" Notation with a Calculator

- a. Numbers in scientific notation are entered into a scientific calculator as follows:
- 1) Enter the significant digits.
  - 2) Press the **E** or **EXP** key. (Actual key label may vary.)
  - 3) Enter the power of 10. If the power is negative press the **+/-** key in conjunction with entering the power.

## J. Order of Mathematical Operations

1. In solving any equation it is necessary to perform the operations in the equation in accordance with a certain hierarchy or order of operations. Equations are solved by simplifying operations of higher order first, according to group, left to right. The order for solving equations is as follows:
  - a. Simplify expressions within grouping symbols, beginning with the innermost set if more than one set is used.
  - b. Simplify all powers.

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- c. Perform all multiplications and divisions in order from left to right.
- d. Perform all additions and subtractions in order from left to right.

2. For example:

$$(3 + 1)^2 \times 3 - 14 \div 2$$

- a. Simplify parentheses.

$$\underbrace{(3 + 1)}^2 \times 3 - 14 \div 2$$

- b. Simplify powers.

$$\underbrace{(4)}^2 \times 3 - 14 \div 2$$

- c. Perform multiplication and division left to right.

$$\underbrace{16 \times 3} - \underbrace{14 \div 2}$$

- d. Perform subtraction.

$$48 - 7 = 41 \text{ (final answer)}$$

#### K. Algebra

1. Algebra is the branch of mathematics which deals with the manipulation of words and letters, generically called symbols, which represent numbers. Two factors contribute to the widespread use of algebra in scientific calculations. First, by using words and letters to represent the values of physical quantities, physical relationships can be expressed in clear, concise, and completely generalized form. Second, by using words and letters in place of numbers, combinations of physical relationships may be simplified to yield results applicable to any set of numbers.
2. For example, the area of a rectangle equals the product of the length of the rectangle multiplied by its width. In generalized terms, this statement can be written as:

$$\text{Area} = \text{Length} \times \text{Width}$$

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3. This expression is a simple rule which tells the relationship between the area and the length and width of a rectangle. It does not mean that words are multiplied together but rather that numbers are inserted for the length and the width to obtain the area. For example, if the length is 4 feet and the width is 2 feet, the area is 2 feet  $\times$  4 feet or 8 square feet. This expression can be further simplified by using symbols or letters instead of words. For example, if area is designated by the letter  $A$ , length designated by the letter  $l$ , and width designated by the letter  $w$ , the following expression results:

$$A = l \times w \text{ or } A = lw$$

4. In algebraic expressions, when two or more letters representing numbers are written next to each other without a symbol between them, multiplication is indicated.
5. Variables vs. Numbers

When words or letters are used to represent numbers, they are called variables. Thus, when letters like  $x$ ,  $y$ ,  $z$ ,  $f$ , or  $k$  are used to represent the values of physical quantities, they are called variables because their value varies with the actual numbers they may be chosen to represent. In the area calculation above,  $A$ ,  $l$ , and  $w$  are variables used to represent the numerical values of area, length and width, respectively.

6. Properties of Variables
- a. Recall that every number has a sign and an exponent associated with it. Recall also that any number can be written as a fraction by putting that number as the numerator and 1 as the denominator, e.g.,  $5 = 5/1$ . These properties also apply to any symbols that we might use to represent numbers. Additionally, a symbol by itself stands for one of whatever the variable represents. That is to say, the symbol  $a$  by itself means "one of the variable represented by the letter  $a$ ", or  $1a$ . Combining this with the other "invisible" properties mentioned, the symbol  $a$  is understood to represent "positive one of the variable represented by the letter  $a$  to the power of one, over 1," which would be expressed as:

$$\frac{+1a^1}{1} = a$$

- b. An expression that is either a numeral, a variable or the product of a numeral and one or more variables is called a monomial. A combination or sum of monomials is called a polynomial. Examples of each are:

1) Monomials

$$12z \quad \text{or} \quad -4x^3$$

2) Polynomials

$$3x + 9 \quad 6a^2 - 15$$

## 7. Equations

- a. An equation is a statement that indicates how two quantities or expressions are equal. The two quantities are written with an equal sign (=) between them. For example,

$$\begin{array}{ll} 1 + 1 = 2 & 10 = 6 - (-4) \\ 5 \times 3 = 15 & 18 \div 2 = 9 \end{array}$$

are all equations because in each case the quantity on the left side is equal to the quantity on the right side.

- b. In algebra we use variables to represent numbers in equations. In this lesson we will manipulate and solve equations involving more than one variable, but we will find the solution, i.e., the final answer, to equations having only one variable.

## 8. Algebraic Manipulation

- a. The basic principle, or axiom, used in solving any equation is: whatever operation is performed on one side of an equation—be it addition, subtraction, multiplication, division, raising to an exponent, taking a root—must also be performed on the other side if the equation is to remain true. This principle must be adhered to in solving all types of equations.
- b. This axiom can be thought of by visualizing the balancing of a scale. If the scale is initially balanced, it will remain balanced if the same weight is added to both sides, if the same weight is removed from both sides, if the weights on both sides are increased by the

same factor, or if the weights on both sides are decreased by the same factor.

- c. Here are the general forms for algebraic manipulation of equations. For the real numbers  $a$ ,  $b$ ,  $c$  and  $n$ :

- 1) Addition

$$\text{If } a = b, \text{ then } a + c = b + c$$

- 2) Subtraction

$$\text{If } a = b, \text{ then } a - c = b - c$$

- 3) Multiplication

$$\text{If } a = b, \text{ then } a \cdot c = b \cdot c$$

- 4) Division

$$\text{If } a = b, \text{ then } a \div c = b \div c$$

- 5) Involution

$$\text{If } a = b, \text{ then } a^n = b^n$$

- 6) Evolution

$$\text{If } a = b, \text{ then } \sqrt[n]{a} = \sqrt[n]{b}$$

See Table 2. - "Rules for Algebraic Manipulation"

Involution - raising to an exponent.

Evolution - taking a root.

## 9. Manipulating and Solving Linear Equations

- a. The addition or subtraction of the same quantity from both sides of an equation may be accomplished by transposing a quantity from one side of the equation to the other. Transposing is a shortened way of applying the addition or subtraction axioms. Any term may be transposed or transferred from one side of an equation to the other if its sign is changed. Thus, in the equation below the +4 can be transposed to the other side of the equation by changing its sign:

$$5x + 4 = 14$$

$$(5x + 4) - 4 = (14) - 4$$

$$5x = 14 - 4$$

$$5x = 10$$

- b. Transposing also works with multiplication and division. Remembering that any number can be expressed as a fraction we can rewrite the last line of the equation above. We can then move the 5 in the numerator of the left side to the denominator of the right side:

$$\frac{5x}{1} = \frac{10}{1}$$

$$\frac{x}{1} = \frac{10}{5}$$

$$x = \frac{10}{5}$$

$$x = 2$$

- c. Some linear equations may contain multiple terms (monomials) involving the same variable. In order to simplify the equation like terms must be combined. Don't forget those "invisible properties of variables." Here's an example:

Solve for  $x$  in the equation:

$$3x - 5 = x + 3$$

- 1) Subtract  $x$  from both sides:

$$3x - 5 - x = x + 3 - x$$

- 2) Combine like terms and cancel:

$$2x - 5 = 3$$

- 3) Add 5 to both sides:

$$2x - 5 + 5 = 3 + 5$$

- 4)  $-5$  and  $+5$  cancel:

$$2x = 8$$

- 5) Divide both sides by 2:

$$\frac{2x}{2} = \frac{8}{2}$$

- 6) 2 over 2 cancels. Reduce.

$$x = 4$$

## 10. Quadratic Equations

- a. In manipulating an equation involving multiple variables, the "variable of interest" (or the variable to be solved for) must be moved to one side of the equal sign and all other variables must be moved to the other side. In order to accomplish this, operations must be performed on both sides of the equation that will result in a variable, or group of variables, to be canceled out from one side. This cancellation can only occur if the "opposite function" is performed on a function that already exists on that side of the equation. This means that a variable that is being multiplied can be canceled by dividing by the same variable. Addition can be canceled with subtraction, multiplication with division, etc.

- b. Example:

Solve for  $a$  in the equation:

$$a + b = c$$

- 1) Subtract  $b$  from both sides:

$$a + b - b = c - b$$

- 2)  $+b$  and  $-b$  cancels, leaving:

$$a = c - b$$

- c. Example:

Solve for  $a$  in the equation:

$$ab = c$$

- 1) Divide both sides by  $b$ :

$$\frac{ab}{b} = \frac{c}{b}$$

- 2)  $b$  over  $b$  cancels, leaving:

$$a = \frac{c}{b}$$

- d. Do not forget that the order of operations must be observed when manipulating equations. Otherwise a completely different solution may result. The key is to do the opposite function in reverse order. Here is an example which shows how this is done.

Solve for  $a$  in the equation:

$$\frac{a + b}{c} = d$$

- 1) Multiply both sides by  $c$ :

$$c \cdot \frac{(a + b)}{c} = d \cdot c$$

- 2)  $c$  over  $c$  cancels:

$$a + b = dc$$

- 3) Subtract  $b$  from both sides:

$$a + b - b = dc - b$$

- 4)  $b - b$  cancels, leaving:

$$a = dc - b$$

- e. Once the order of the arithmetic functions has been established, manipulation of the formula can begin. In the example above, if the values for  $a$ ,  $b$ , and  $c$  were known, the first step would be to add  $b$  to  $a$ . The second step would be to divide by  $c$ . Therefore, in order to solve it, we do the opposite functions in reverse order. So, we first multiply by  $c$ . Then, we would subtract  $b$ . It is a good idea to rewrite the equation each time so that the operations can be reevaluated before the next step. One final example:

Solve for  $d$  in the equation:

$$ab^2 = cd^2$$

- 1) Divide both sides by  $c$ :

$$\frac{ab^2}{c} = \frac{cd^2}{c}$$

- 2)  $c$  over  $c$  cancels:

$$\frac{ab^2}{c} = d^2$$

- 3) Take the square root of both sides:

$$\sqrt{\frac{ab^2}{c}} = \sqrt{d^2}$$

- 4) Square root of square cancels, leaving:

$$\sqrt{\frac{ab^2}{c}} = d$$

## 11. Substitution

Linear equations are solved by combining like terms and reducing to find the solution. Quadratic equations are solved by substituting given values into the equation for all but one of the variables, thus making it a linear equation. The best approach is to first solve for the variable of interest by algebraic manipulation. Then find the solution by substituting the given values into the equation for the respective variables. The single, unknown variable, or the variable of interest, will be left on one side, being set equal to the solution. For example:

Given the equation:

$$2x - y^2 = 3a - b$$

Where  $x = 5$ ,  $y = (-4)$  and  $a = 3$ ; solve for  $b$

- a. To solve for  $b$ :

$$2x - y^2 = 3a - b$$

- b. Add  $b$  to both sides:

$$2x - y^2 + b = 3a - b + b$$

- c.  $-b + b$  cancels:

$$2x - y^2 + b = 3a$$

- d. Transpose  $2x - y^2$  to right side:

$$b = 3a - 2x + y^2$$

e. Substitute known values:

$$b = 3(3) - 2(5) + (-4)^2$$

f. Perform operations:

$$b = 9 - 10 + 16$$

g. Simplify:

$$b = 15$$

#### L. Logarithms

1. In many cases, arithmetic operations can be performed much more quickly if the numbers involved are numbers with exponents to the same base. For example, the multiplication or division of numbers with exponents to the same base can be performed by merely adding or subtracting the exponents. Raising to a power or taking a root can be performed by merely multiplying or dividing the exponents by the power or root. It is this feature of numbers with exponents which led to the development of logarithms. If all numbers could be readily written as numbers with exponents to the same base, multiplication, division, raising to powers and taking roots could be performed much more quickly.
2. Any number can be expressed as a power of any other number. For example, 64 equals  $2^6$ ,  $4^3$ , or  $8^2$ . 64 also equals  $7^{2.137}$  or  $10^{1.806}$ . The use of logarithms involves expressing numbers as powers of a common number, such as 10, so that arithmetic operations with these numbers can be performed more quickly.
3. Simply put, a logarithm is an exponent. More explicitly, the logarithm of a number ( $n$ ) is the exponent ( $x$ ) of a given base ( $B$ ) that is required to produce that number. The symbol  $\log$  is used to denote taking a logarithm. The base is usually indicated by a small number written to the right and slightly below the symbol  $\log$ . The general relationship and form are as follows:

$$\text{If } n = B^x;$$

where  $B > 0$  and  $B \neq 1$ ;

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then:  $\log_b n = x$

4. For example:

$$1000 = 10^3 \Rightarrow \log_{10} 1000 = 3$$

5. This says that the base ten logarithm of 1000 is 3, which means that the base number, 10, must be raised to the power of 3 to equal 1000.

6. Here are some additional examples:

$$2^3 = 8 \Rightarrow \log_2 8 = 3$$

$$4^{3/2} = 8 \Rightarrow \log_4 8 = 3/2$$

7. Before the development of the pocket calculator, the use of logarithms saved considerable computation time. For example, the evaluation of the following expression by hand would take a very long time.

$$\frac{(865)^3(0.0247)^{1/2}(93,800)}{(0.00186)(4.53)^2}$$

8. However, using logarithms the above expression could be evaluated in a matter of minutes. Thus, logarithms, or logs, became one of the most useful tools in mathematics. In addition to simplifying arithmetic calculations and shortening computation time, logs are also important in engineering applications. The relationship between a number and its logarithm is used frequently to assist in measuring physical quantities when they vary over a wide range. For example, logarithmic scales are used to measure the neutron flux in nuclear reactors. Logarithms are also used for scales on charts and meters.

9. Properties of Logarithms

- a. Since logarithms are exponents, the basic rules of exponents can be used to develop several useful properties of logarithms. Suppose that  $a$ ,  $x$ , and  $y$  are numbers, and  $a$  is a suitable base for a logarithm ( $a > 0$ ,  $a \neq 1$ ). The product rule for exponents says:

$$a^x \cdot a^y = a^{(x+y)}$$

- b. Let us say that:

$$u = a^x \quad \text{and} \quad v = a^y$$

- c. If we write each of these in logarithmic form we would have:

$$x = \log_a u \quad \text{and}$$

$$y = \log_a v$$

- d. Then:  $u \cdot v = a^x \cdot a^y = a^{(x+y)}$

- e. If we write this in logarithmic form it would be:

$$\log_a (u \cdot v) = x + y$$

- f. If we substitute the values for  $x$  and  $y$  from above we have:

$$\log_a (u \cdot v) = \log_a u + \log_a v$$

- g. This results in one of the rules for logarithms, the product rule. Using similar methods, we could also prove the other two rules that have been developed for logarithms.

See Table 3 - "Rules for Logarithms"

#### 10. Base Ten Logarithms

- a. Logs with the base of 10 are the most commonly used logarithms because of their relationship to the place values in the decimal system. Because of their wide use, base ten logarithms are often referred to as common logarithms.
- b. Observe the patterns in the number line and table. Notice the relationship between the power of ten and the logarithm.
- c. Any number can be expressed as a power of ten. Thus, 10 equals  $10^1$ , 1,000 equals  $10^3$ , 64 equals  $10^{1.806}$  and 527.3 equals  $10^{2.722}$ . Once a number has been expressed as a power of ten, the base ten logarithm of the number is known—it is the exponent of 10. Thus  $\log_{10} 10$  equals 1,  $\log_{10} 1000$  equals 3,  $\log_{10} 64$  equals 1.806 and  $\log_{10} 527.3$  equals 2.722. Since base ten logarithms are so commonly used, the subscript 10 is

Refer students to number of line showing powers of 10 and Table 4 - "Log to Power of 10 Relationship."

often omitted after the symbol log. Thus,  $\log 27.3$  means the logarithm of 27.3 to the base 10.

- d. A common logarithm is most often a mixed number consisting of a whole number part and a decimal fraction part. The whole number part is called the characteristic of the logarithm. The decimal fraction part is called the mantissa. For example, in the logarithm of 527.3, which equals 2.722, the characteristic is 2 and the mantissa is 0.722. The mantissas of most logarithms are rounded off to a specified number of significant digits. Typically, mantissas are rounded off to three, four, five or more significant digits.

#### 11. Log-Table Method

- a. To find the common logarithm of a number using four-place log tables, use the following steps:
- 1) Write the number in scientific notation with up to four significant digits.
  - 2) Using the product rule for logarithms, write the log of the product as the sum of the logs of the factors.
  - 3) Determine the mantissa as follows:
    - a) Find the row in the table corresponding to the first two digits of the number, then move over to the column corresponding to the third digit of the number.
    - b) Find the number under the proportional parts section corresponding to the fourth digit of the number, and add it to the last digit of the decimal obtained in step 3.a.
  - 4) Determine the characteristic by using the power of 10 written in step 1 and write the logs in the form of a sum.

NOTE: If the power of 10 is negative, the log may be left in this form.

See Table 5 - "Four-place Logarithms"

NOTE: If the power of 10 is negative, the mantissa will be changed because of the subtraction of a whole number.

- b. As you may have observed, the mantissa of the base ten logarithm of a number depends only on the succession of significant digits in the number. The position of the decimal point in the number does not affect the mantissa. Of course, the characteristics are different for each of these numbers.
- c. Note that any time the logarithm of a number is rounded off it would be considered an approximate answer since each digit is necessary to exactly duplicate the number when the base is raised to that exponent. Since each significant digit of the logarithm affects the actual value of the number, a standard of four significant digits should be maintained to ensure appropriate accuracy in the answers.

#### 12. Calculator Method

Since hand-held scientific calculators are readily available today, it is impractical to use log tables. To find the logarithm of a number with a calculator:

- a. Enter the number.
- b. Press the  $\boxed{\log}$  key. The number displayed is the logarithm of the number entered in step 1.

#### 13. Natural Logarithms

- a. A logarithm can be written to any base. For most practical computations, base ten logarithms are used because of their relationship to the place values in the decimal system. However, in many scientific and engineering problems, it is convenient to use another base, symbolized by the letter  $e$ .  $e$  is an irrational number whose value is 2.71828 . . . The actual value of  $e$  is the limiting value of  $(1 + 1/n)^n$  as  $n$  gets larger and larger.
- b. Although it is an irrational number, it can still be used as the base for logarithms in the same way as 10 is used for base ten logarithms.  $e$  is the basis for many laws of nature, such as the laws of growth and decay of physical quantities, including the decay of radioactive

Do examples in the study guide.

See Table 6 - "Mantissa for Successive Significant Digits."

substances and the growth and decay of neutron population in a nuclear reactor. Because of the relationship of  $e$  to natural phenomena, logarithms to the base  $e$  are called natural logarithms.

- c. The natural logarithm of a number is the exponent to which  $e$  must be raised in order to get that number. The symbol  $\ln$  is used to denote a natural logarithm which is the same as saying  $\log_e$ . The relationship is expressed as follows:

$$\text{If } e^x = n \text{ then } \ln n = x$$

- d. For example:

1)  $\ln 2 = 0.693147$

which means that  $e^{0.693147\dots} = 2$ .

2)  $\ln 10 = 2.302585$

which means that  $e^{2.302585\dots} = 10$ .

3)  $\ln e = 1$

which means that  $e^1 = e$

- e. Natural logarithms are not often used for computations. However, they appear frequently in decay and shielding calculations problems because of the relationship of  $e$  to natural phenomena. As a result, it is important to know how to determine the natural logarithms of numbers.
- f. Tables of natural logarithms are available in several standard handbooks. However, there are several important differences between natural logarithms and base ten logarithms which must be understood to use natural logarithms. A natural logarithm is not separated into a characteristic and a mantissa. This is because the whole number part of a natural logarithm does not relate to the position of the decimal point. Therefore, tables of natural logarithms give the entire logarithm, not just the decimal fraction part. Moreover, if a natural logarithm is negative, the entire logarithm is negative and is shown as such in a table of natural logarithms. Further, there is no part of the natural logarithm of a number which is not affected by

the position of the decimal point. For all these reasons, tables of natural logarithms cannot be made concise.

- g. To find the natural log of a number using a hand-held calculator:

- 1) Enter the number.
- 2) Press the  $\boxed{\ln}$  key. The number displayed is the natural logarithm of the number entered in step 1.

#### 14. Antilogarithms

- a. An antilogarithm, usually shortened to "antilog," is the opposite of a logarithm and is much easier to do. The antilog of a given number is the value obtained by raising the base to that number. Finding antilogs is an important part in the overall use of logarithms in computations. If numbers are converted to logarithms to perform calculations, the answer must be converted back from logarithms once the calculations have been performed. The symbol  $\log^{-1}$  is used in calculations to indicate the antilog is going to be taken. The base of 10 is assumed unless otherwise noted. The general form is:

$$\log^{-1} x = n \text{ which means } 10^x = n$$

- b. For example:

$$\log^{-1} 3 \text{ which means } 10^3 = 1000$$

- c. On a scientific calculator the base 10 antilog of a number is obtained by raising the base (10) to that number.
- 1) Enter the number.
  - 2) Press the  $\boxed{\log^{-1}}$  or  $\boxed{10^x}$  key. The number displayed is the antilog of the number of entered in step 1. In other words, 10 raised to that power.
- d. The symbol  $\ln^{-1}$  is used to denote the inverse natural log, i.e. the antilog of base  $e$ .

$$\ln^{-1} x = n \text{ which means } e^x = n$$

Refer also to example in study guide.

- e. For example:

$$\ln^{-1} 0.693 \text{ which means } e^{0.693\dots} = 2$$

- f. On a scientific calculator the inverse natural log of a number is obtained as follows:

- 1) Enter the number.
- 2) Press the  $\boxed{\ln^{-1}}$  or  $\boxed{e^x}$  key. The number displayed is the inverse natural log of the number of entered in step 1. In other words,  $e$  raised to that power.

See Table 7 - "Summary of Log Definitions."

#### 14. Solving for Variables as Exponents

- a. One of the useful applications for logarithms is to solve algebraic equations with unknown exponents. In the following example, for instance, if the exponent is not known, it would be difficult to determine the correct value of  $x$  in order to make the statement (or equation) true.

$$2356 = 3^x$$

- b. With the use of logarithms, however, this type of problem can be easily solved. The steps for solving an equation of this type are:

- 1) Make sure the base raised to the unknown exponent is isolated on one side of the equation (this may involve some manipulation of the formula in more complicated equations).

- 2) Take the log of both sides of the equation:

$$\log 2356 = \log 3^x$$

- 3) Since a logarithm is an exponent, and this is an exponent raised to a power, this statement can now be rewritten using the laws of exponents, which say that the exponent and power are multiplied:

$$\log 2356 = (\log 3)(x)$$

- 4) Divide both sides by  $\log 3$  which moves it to the right side of the equation:

$$\frac{\log 2356}{\log 3} = \frac{(\log 3)(x)}{\log 3}$$

- 5) Cancel terms and rewrite the equation:

$$\frac{\log 2356}{\log 3} = x$$

- 6) Perform the operations and solve:

$$\begin{aligned} \frac{3.372}{0.477} &= x \\ 7.068 &= x \end{aligned}$$

- c. This answer can now be checked by substituting it back into the original equation to see if it makes the statement true:

$$2356 = 3^{7.068}$$

- d. Some problems may involve the base of the natural logarithm,  $e$ , raised to an unknown power. This exponent can be determined by isolating  $e$  on one side of the equation and then taking the natural log of both sides. This is done because taking the natural log of  $e$  reduces to 1. For example:

$$\begin{aligned} 125 &= 1000e^n \\ \frac{125}{1000} &= e^n \\ \ln \frac{125}{1000} &= \ln e^n \\ \ln 0.125 &= (\ln e)(n) \\ \ln 0.125 &= (1)(n) \\ -2.0794 &= n \end{aligned}$$

### III. SUMMARY

#### A. Review major topics

1. Fractions
2. Decimals

3. Percent
4. Signed Numbers
5. Exponents
6. Square Roots
7. Scientific Notation
8. Order of Mathematical Operations
9. Algebra
10. Logarithms

B. Review learning objectives

#### **IV. EVALUATION**

Evaluation should consist of a written examination comprised of multiple choice questions. 80% should be the minimum passing criteria for the examination.

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